JOURNAL OF APPROXIMATION THEORY 41, 361–366 (1984)

On Trigonometric n-Widths and Their Generalization

Y. MAKOVOZ

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078 U.S.A.

Communicated by Charles A. Micchelli

Received May 24, 1983

Let Ω be a compact set in a Banach space X. The quantity

$$d_n(\Omega, X) = \inf_{\Gamma_n} \sup_{x \in \Omega} \inf_{\gamma \in \Gamma_n} \|x - \gamma\| \tag{1}$$

is called the n-width of Ω in X (in the sense of Kolmogorov). The left infimum in (1) is taken over *all* n-dimensional linear subspaces $\Gamma_n \subset X$. One can obtain various modifications of this definition by taking the infimum over special classes of Γ_n. For instance, in $X = L_q[0, 2\pi]$ one may consider subspaces Γ_n spanned by any n of the functions $\{\exp(ik\,\cdot\,)\}$, $k \in \mathbb{Z}$. If the infimum in (1) is taken over all such subspaces, the corresponding n-width, introduced by Ismagilov [1], is called the trigonometric n-width, $d_n^T(\Omega, L_q)$. It is obvious that $d_n^T \geqslant d_n$, but if the class Ω is translation-invariant, one would expect that $d_n^T = d_n$. And indeed, in all cases for which d_n^T has been estimated, $d_n^T \sim d_n (n \to \infty)$.

In a more general setting, let G be a compact Abelian group with the invariant measure μ, $\mu(G) = 1$. In $X = L_q = L_q(G, \mu)$ we consider the subspaces Γ_n of the form

$$\Gamma_n = \operatorname*{span}_{1 \leqslant v \leqslant n} \{\chi_v\}, \tag{2}$$

where χ_v are the (continuous) characters of G. The n-width (1), with $X = L_q$ and the infimum taken over all Γ_n of the form (2), will be denoted by $d_n^G(\Omega, L_q)$.

Let now φ be a complex-valued function on G represented in $L_q(G, \mu)$ by the series

$$\varphi = \sum_{k=1}^{\infty} a_k \chi_k, \tag{3}$$

where $\{\chi_k\}$ is a sequence of (not necessarily all) characters and a_k are

0021-9045/84 $3.00

complex numbers. Let $W_p(\varphi)$ be the following class of complex-valued functions

$$W_p(\varphi) = \{x : x = \varphi * z,\ \|z\|_p \leqslant 1\} \tag{4}$$

(here and below $\|\cdot\|_p = \|\cdot\|_{L_p}$). We shall assume for simplicity that $|a_k|$ are nonincreasing. Let also

$$C_0 = \sup_n \inf_m \{n\,|a_n| \cdot A_{nm}^{-1}\} < \infty, \tag{5}$$

where $A_{nm} = |a_n| + \cdots + |a_m|$. The condition (5) is obviously satisfied if $\sum |a_k| = \infty$. It follows from (5) that for every n there exists such $m > n$ that

$$Cn\,|a_n| \leqslant A_{nm}, \tag{6}$$

where $C = 1$ (if $\sum |a_k| = \infty$) or $C = (2C_0)^{-1}$.

We shall use the notation $\alpha_n \preccurlyeq \beta_n$, if for two sequences, $\{\alpha_n\}$ and $\{\beta_n\}$, there is a constant M, independent of n, such that $\alpha_n \leqslant M\beta_n$ for all n. We shall write $\alpha_n \asymp \beta_n$ if both $\alpha_n \preccurlyeq \beta_n$ and $\beta_n \preccurlyeq \alpha_n$. Our main result is the following statement.

THEOREM 1. *Let $W_p(\varphi)$ be the class of functions defined by* (3) *and* (4) *with $|a_k|$ nonincreasing and satisfying* (5).

(a) *For $2 \leqslant q < \infty$,*

$$d^G_{c_1 n}(W_1(\varphi), L_q) \preccurlyeq A_{nm} n^{-1/2} + R_{mq}, \tag{7}$$

where m is any number satisfying (6), $C_1 = 1 + 2C$ *and*

$$R_{mq} = \left\| \sum_{k=m+1}^{\infty} a_k \chi_k \right\|_q .$$

(b) *If, in addition, $\sum |a_k| < \infty$, $1 \leqslant p \leqslant 2 \leqslant q < \infty$, $p^{-1} + q^{-1} > 1$, then*

$$d^G_n(W_p(\varphi), L_q) \preccurlyeq A_{n\infty} \cdot n^{1/p - 3/2}. \tag{8}$$

Let now for $p \geqslant 1$ and $r > 0$, W^r_p be the class of complex-valued functions on $T = R/2\pi\mathbb{Z}$ with the rth derivative (in the sense of Weyl) restricted by the inequality $\|x^{(r)}\|_p \leqslant 1$. This class if compact in L_q if $r > p^{-1} - q^{-1}$. As an application of Theorem 1 we prove

THEOREM 2. *Let $1 \leqslant p \leqslant 2 \leqslant q < \infty$, $p^{-1} + q^{-1} > 1$.*

(a) *For* $r > 1$

$$d_n^T(W_p^r, L_q) \asymp n^{-r+1/p-1/2}. \tag{9}$$

(b) *For* $1 - q^{-1} < r < 1$

$$d_n^T(W_1^r, L_q) \asymp n^{(q/2)(1-r)-1/2}. \tag{10}$$

For unrestricted (non-trigonometric) n-widths, (9) and (10) are known from [2] and [3], respectively. In the trigonometric case, (9) was established by V. E. Maiorov [4] for $r > 1/p + 1/2$. His rather complicated proof was based on specific properties of the trigonometric system.

To prove Theorem 1 we need two lemmas.

LEMMA 1. (Rosenthal [5]). *Let* $2 \leqslant q < \infty$. *Then there exists a constant* K_q *depending only on* q *so that if* $\gamma_1, \gamma_2, ..., \gamma_n$ *are independent random variables belonging to* L_q *and* $E\{\gamma_k\} = 0$ *for all* k, *then*

$$\left(E\left\{\left|\sum_{k=1}^{m} \gamma_k\right|^q\right\}\right)^{1/q} \leqslant K_q \max\left\{\left(\sum_{k=1}^{m} E\{|\gamma_k|^q\}\right)^{1/q}, \left(\sum_{m=1}^{m} E\{|\gamma_k|^2\}\right)^{1/2}\right\}.$$

LEMMA 2. *Let* $Q = \{t\}$ *be a set with a measure* μ; $2 \leqslant q < \infty$; $u_k \in L_q(Q, \mu)$ $(k = 1, 2, ...)$ *and* $\sup_k |u_k(t)| = S(t) \in L_q$. *Let* u *be a function representable in* L_q *by the series*

$$u = \sum_{k=1}^{\infty} a_k u_k,$$

where a_k *are complex numbers with* $|a_k|$ *nonincreasing and satisfying* (5). *Then for any natural* n *there exists* $v^* \in L_q$ *of the form*

$$v^* = \sum_{k=1}^{\infty} b_k^* u_k$$

with at most $c_1 n$ *non-zero* b_k^* *such that*

$$\|u - v^*\| \leqslant A_{nm} n^{-1/2} + R_{mq}, \tag{11}$$

where m *is any number satisfying* (6), $c_1 = 1 + 2C$, $R_{mq} = \|\sum_{k=m+1}^{\infty} a_k u_k\|_q$.

Proof. For a given n, let m be a number satisfying (6). We define independent random variables $b_n, b_{n+1}, ..., b_m$ by the formula

$$b_k = \begin{cases} a_k \theta_k^{-1}, & \text{with the probability } \theta_k = C_n |a_k| A_{nm}^{-1} \\ 0, & \text{with the probability } 1 - \theta_k. \end{cases}$$

We shall estimate the expectation $E\{\|u-v\|_q\}$, where v is defined by

$$v=\sum_{k=1}^{n-1} a_k u_k + \sum_{k=n}^{m} b_k u_k .$$

We have $\|u-v\|_q \leqslant R_{mq} + \|w\|_q$, where

$$w=\sum_{k=n}^{m} (a_k-b_k)\, u_k .$$

Now observe that

$$E\{\|w\|_q^q\} = \int_Q E\left\{\left|\sum_{k=n}^{m} (a_k-b_k)\, u_k(t)\right|^q\right\} d\mu. \tag{12}$$

Set, for a fixed t, $\gamma_k=(a_k-b_k)\, u_k(t)$. We wish to apply Lemma 1 to γ_k. Obviously, $E\{\gamma_k\}=0$. We also have

$$\begin{aligned} E\{|\gamma_k|^q\} &= |u_k(t)|^q\, E\{|a_k-b_k|^q\} \\ &\leqslant S^q(t)\cdot |a_k|^q\, [(1-\theta_k)^q\, \theta_k^{1-q} + 1-\theta_k] \\ &\leqslant 2S^q(t)\cdot |a_k|^q\, \theta_k^{1-q}. \end{aligned}$$

So

$$\begin{aligned} \sum_{k=n}^{m} E\{|\gamma_k|^q\} &\leqslant 2S^q(t) \sum |a_k|^q\, [Cn\, |a_k|\cdot A_{nm}^{-1}]^{1-q} \\ &= 2C^{1-q}\cdot S^q(t)\, n^{1-q}\cdot A_{nm}^q. \end{aligned} \tag{13}$$

Hence,

$$\left(\sum_{k=n}^{m} E\{|\gamma_k|^2\}\right)^{q/2} \leqslant 2^{q/2} C^{-q/2} S^q(t)\, n^{-q/2} A_{nm}^q. \tag{14}$$

Comparing (13) and (14) and applying Lemma 1 to (12), we have

$$E\{\|w\|_q^q\} \lesssim A_{nm}^q\, n^{-q/2} \int_Q S^q(t)\, dt \asymp A_{nm}^q\, n^{-q/2}. \tag{15}$$

On the other hand, the expectation of the number $\tilde{n}$ of non-zero coefficients $\{b_k\}$ is

$$E\{\tilde{n}\} = \theta_n + \cdots + \theta_m = Cn.$$

Since both random variables, $\|w\|_q^q$ and $\tilde{n}$, are positive, there should exist a realization for which both $\|w\|_q^q \leqslant 2E\{\|w\|_q^q\}$ and $\tilde{n} \leqslant 2E\{\tilde{n}\}$. Let us denote

b_k, w and v corresponding to this realization by b_k^*, w^* and v^*, respectively. It follows from (15) that, in view of the preceding remark, $\|u - v^*\|_q \leqslant R_{mq} + \|w^*\|_q \lesssim R_{mq} + A_{nm} n^{-1/2}$, which proves the lemma since the number of nonzero coefficients in v^* does not exceed $n + 2Cn$.

Proof of Theorem 1. Since $|\chi_k| = 1$, Lemma 2 is applicalbe with $u = \varphi$ and $u_k = \chi_k$. Let ψ denote v^* provided by Lemma 2. Now, for $x \in W_1(\varphi)$, $x = \varphi * z$, $\|z\|_1 \leqslant 1$, set $y_x = \psi * z$. By the Young inequality for convolutions,

$$\|x - y\|_q \leqslant \|\varphi - \psi\|_q \cdot \|z\|_1 \lesssim A_{nm} \cdot n^{-1/2} + R_{mq}, \tag{16}$$

while $\{y_x\}$ belongs to a subspace spanned by at most $C_1 \cdot n$ characters, so (7) is proved.

Now consider the operator

$$Kz = (\varphi - \psi) * z.$$

To prove (8) it is enough to show that

$$\|K: L_p \to L_q\| \lesssim A_{n\infty} \cdot n^{1/p - 3/2}. \tag{17}$$

For $p = 1$, $q \geqslant 2$ this is a limit $(m \to \infty)$ version of (16). On the other hand, it is easy to verify that, in view of (5), in the notation of Lemma 2 (with $v^* = \psi^*$) $\|K: L_2 \to L_2\| \leqslant \sup_k |a_k - b_k^*| \leqslant 2A_{n\infty}(Cn)^{-1}$, which agrees with (17). Now, for the indicated range (p, q), (17) is an immediate consequence of the Riesz–Thorin interpolation theorem. This finishes the proof of (8) since the distinction between n and $C_1 n$ is, in the context of (8), immaterial.

Proof of Theorem 2. If $x \in W_p^r$, then $x = \text{const} + \varphi * z$, where $\|z\|_p \leqslant 1$,

$$\varphi = \sum_{k \neq 0} (ik)^{-r} \exp(ik \cdot).$$

The conditions of Theorem 1 are obviously fulfilled. If $r > 1$, then $A_{n\infty} \asymp n^{-r+1}$ and the upper estimate in (9) follows from (8).

If $1 - q^{-1} < r < 1$, we set $m = n^{q/2}$ in (7). It is easy to verify (using, for instance, Abel's transformation) that

$$\left\| \sum_{|k| > m} (ik)^{-r} \exp(ik \cdot) \right\|_q \lesssim m^{-r+1-q^{-1}}.$$

So, by (7)

$$d_{c_1 n}^T(W_1^r, L_q) \lesssim m^{-r+1} \cdot n^{-1/2} + m^{-r+1-q^{-1}} \asymp n^{(q/2)(1-r)-1/2}$$

which is equivalent to the upper estimate in (10).

The lower estimates in (9) and (10) follow from [2] and [3].

Remarks. (1) The condition $|a_k| \downarrow$ in Theorem 1 is not essential for it can be always satisfied by rearrangement. Indeed, for both the statement and the proof of (7) this condition is needed only for a finite number of terms, while in the case (b) the series (3) converges unconditionally. This observation is important when there is no "natural" order of characters in (3). To illustrate this remark, consider the class W defined by (3) and (4) in the case when $G = T^2$, $(t, t_1) \in T^2$,

$$\varphi(t, t_1) = \sum_{k, k_1 \neq 0}' (ik)^{-r} (ik_1)^{-r_1} \exp(ikt + ik_1 t_1). \tag{18}$$

We assume here that $1 < r \leqslant r_1$. Let a_j denote the absolute values of the coefficients in (18) arranged in a nonincreasing way. An elementary computation shows that $a_j \asymp j^{-r}$ if $r < r_1$ and $a_j \asymp j^{-r}(\log j)^r$ if $r = r_1$. The conditions of Theorem 1(b) are satisfied and (8) gives

$$d_n^T(W, L_q) \ll n^{-r+1/p-1/2} \cdot (\log n)^{r\alpha}, \tag{19}$$

where $\alpha = 0$ if $r < r_1$ and $\alpha = 1$ if $r = r_1$. This estimate for the unrestricted (non-trigonometric) case was announded in [6], where it is also claimed that the order in (19) is exact for d_n and therefore, for d_n^T.

(2) The order of $d_n(W_1^1, L_q)$ $(2 < q < \infty)$ is still not known: the upper estimate provided by Theorem 1 and the lower estimate from [3] differ in a logarithmic factor.

(3) The method of approximation considered in this paper is linear. So Theorem 2 gives also an estimate for the linear n-widths, $a_n(W_p^r, L_q)$. Our proof can be extended to give an exact order of a_n also for the case $r > 1$, $p \leqslant 2 \leqslant q$, $p^{-1} + q^{-1} < 1$. However, in terms of linear n-widths, our result is new only for $r < 1$ (see, e.g. [7]).

References

1. R. S. Ismagilov, Diameters of sets in normed linear spaces and the approximation of functions by trigonometric polynomials, *Uspekhi Mat. Nauk* **29**, No. 3 (1974), 161–178; *Russian Math. Surveys* **29**, No. 3 (1974), 169–186.
2. B. S. Kashin, The widths of certain finite dimensional sets and classes of smooth functions, *Izv. Akad. Nauk SSSR* **41** (1977), 334–351; *Math. USSR Izv.* **11** No. 2 (1977).
3. B. S. Kashin, Diameters of Sobolev classes of small order smoothness, *Vestnik Moskov. Univ.* Math. **5** (1981), 50–54. [Russian]
4. V. E. Maiorov, On linear widths of Sobolev classes and chains of extremal subspaces, *Mat. Sb.* **113** (155), 3; *Math. USSR Sb.* **41** No. 3 (1982), 361–381.
5. H. Rosenthal, On the subspaces of $L^p(p > 2)$ spanned by sequences of independent random variables, *Israel J. Math.* **8** (1970), 273–303.
6. V. N. Temlyakov, Widths of certain classes of functions of several variables, *Dokl. Akad. Nauk SSSR* **267**, No. 2 (1982), 314–317. [Russian]
7. K. Höllig, Approximationszahlen von Sobolev Einbettungen, *Math. Ann.* **242** (1979), 273–281.